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Numerical analysis for semilinear evolution equations of parabolic type

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Abstract

We study the Galerkin Euler approximations of semilinear evolution equations of parabolic type. We utilize both the semigroup method and the variational method to construct approximate solutions and estimate errors.

Keywords: Semilinear abstract evolution equations of parabolic type; Galerkin method; Implicit–explicit Euler scheme; Semigroup method; Variational method

1. Introduction

We study the full discretization problem for the semilinear abstract evolution equation

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + AU = F(U), \quad 0 < t < T, \\ U(0) = U_0 \end{cases} \tag{1.1}$$

of parabolic type in certain abstract space to be defined below. We will consider the Galerkin Euler approximation of (1.1), and then discuss the construction of the approximate solutions and the estimation of errors in two ways: one is the semigroup method, the method employing the semigroup of linear operators; the other the variational method, the method of energy estimation.

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The semigroup method is, as is well-known, a very powerful technique [12,13]. The authors have studied in [5–8] a full discretization for parabolic equations in Banach spaces, utilizing the method. There we have obtained the error estimate of approximate solution in the maximum norm. On the other hand, we know also that the technique is very complicated because of the long procedure for constructing evolution operators. For the method, we refer the reader to Fujita et al. [1] and the references therein.

On the contrary, the variational method is a very simple technique [3]. Using the energy inequality, we can obtain a priori estimates of the norms and the lifespan of solutions. We shall see that applications of the method to approximate equations yield almost the same estimates for its solutions and its error. However, the variational method requires that the underlying space is a Hilbert space, and that the operator A be self-adjoint. This is the only difficulties of the method. For the method, we refer the reader to Thomée [14] and the references therein.

The objective of this article is to compare the abstract results obtained by the semigroup method and the ones by the variational method. Section 2 is devoted to applying the semigroup method to verify the similar results shown in [5,8]. In Section 3 the main results by the variational method are proved. Finally in Section 4 we will give some remarks on the difference between these two methods.

In this article the application of these theories to the practical systems are omitted. For example, the authors have studied in [7,8] the full discretization of the chemotaxis system [2,4]

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u - \nabla\{u\nabla\chi(v)\} + f(u), \\ \dfrac{\partial v}{\partial t} = b\Delta v + cu - dv. \end{cases} \tag{1.2}$$

In those papers we have formulated (1.2) as a quasilinear equation $\mathrm{d}U/\mathrm{d}t + A(U)U = F(U)$ with positive non-monotone operators $A(U)$ in the product L^2-space. On the other hand, we can also handle (1.2) as a semilinear equation (1.1) with self-adjoint A in the product Hilbert space $L^2 \times H^s$, $s > 0$. In this case, the variational method is applicable as well as the semigroup method. However, the application of the conforming finite element methods in such a Sobolev space requires the use of finite elements of $\mathscr{C}^1$-class or higher, and basic estimates of such elements. They will be studied in the forthcoming papers.

2. Semigroup approach

Consider the Cauchy problem of a semilinear equation

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + AU = F(U), \quad 0 < t < T, \\ U(0) = U_0 \end{cases} \tag{2.1}$$

in a Banach space X. Here, A is the negative generator of an analytic semigroup on X with the domain $\mathscr{D} = \mathscr{D}(A)$. $F(U)$ is a continuous operator from $Z = \mathscr{D}(A^\beta)$ to X and $\beta \in [0, 1)$ is some exponent. $U_0 \in Z$ is an initial value. $U = U(t)$ is the unknown function.

We make the following assumptions:

(A) $\rho(A)$ contains $\mathbb{C} \setminus \overline{S_\varphi}$, $0 < \varphi < \pi/2$, and the resolvent satisfies

$$\|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leqslant \frac{M_A}{|\lambda| + 1}, \quad \lambda \notin \overline{S_\varphi}$$

with some constant M_A, where $S_\varphi = \{z \in \mathbb{C}; |z| > 0, \ |\arg z| < \varphi\}$ is a sectorial domain.

(F) $F(\cdot)$ satisfies

$$\|F(U) - F(V)\|_X \leqslant p(\|U\|_Z + \|V\|_Z)\|U - V\|_Z, \quad U, V \in Z$$

with some continuous increasing function $p(\cdot)$.

(In) U_0 is in $\mathscr{D}$.

We can easily verify that, under assumptions (A) and (F), Eq. (2.1) possesses a unique local solution

$$U \in \mathscr{C}([0, T_{U_0}]; Z) \cap \mathscr{C}^1((0, T_{U_0}]; X) \cap \mathscr{C}((0, T_{U_0}]; \mathscr{D}),$$

where $T_{U_0} > 0$ is a constant determined by $\|U_0\|_Z$. We refer the reader to [15,16] for example.

Let $\{X_\xi\}_{\xi > 0}$ be a family of finite-dimensional subspaces of X such that $X_\xi \subset Z$. Denote by Z_ξ the space X_ξ equipped with the induced norm of Z. For $\xi > 0$, $P_\xi : X \to X_\xi$ is a projection operator; and, as $\xi \to 0$, $P_\xi \to I$ strongly on X. Let A_ξ be an approximate operator of A such that A_ξ is a bounded linear operator on X_ξ. Then the approximate equation in X_ξ is given by

$$\begin{cases} \dfrac{\mathrm{d}\hat{U}}{\mathrm{d}t} + A_\xi \hat{U} = F_\xi(\hat{U}), \quad 0 < t < T, \\ \hat{U}(0) = P_\xi U_0, \end{cases} \tag{2.2}$$

where $F_\xi(U) = P_\xi F(U)$.

On (2.2) we assume the following conditions:

($\mathrm{A}_\xi 1$) $\rho(A_\xi)$ contains $\mathbb{C} \setminus \overline{S_{\hat{\varphi}}}$, $0 < \hat{\varphi} < \pi/2$, and the resolvents satisfy

$$\|(\lambda - A_\xi)^{-1}\|_{\mathscr{L}(X_\xi)} \leqslant \frac{\hat{M}_A}{|\lambda| + 1}, \quad \lambda \notin \overline{S_{\hat{\varphi}}}$$

with some constant $\hat{M}_A$ independent of ξ.

($\mathrm{A}_\xi 2$) The norm of A_ξ is estimated by

$$\|A_\xi\|_{\mathscr{L}(X_\xi)} \leqslant \hat{N}_A Q_\xi^{-1}$$

with some constant $\hat{N}_A$ independent of ξ, where Q_ξ denotes a function of ξ such that $Q_\xi \to 0$ as $\xi \to 0$.

($\mathrm{A}_\xi 3$) The operator $R_\xi = A_\xi^{-1} P_\xi A$ satisfies

$$\|(1 - R_\xi)A^{-1}\|_{\mathscr{L}(X)} \leqslant \hat{M}_R Q_\xi$$

with some constant $\hat{M}_R$ independent of ξ.

($\mathrm{Sp}_\xi 1$) The norms $\|P_\xi\|_{\mathscr{L}(X)}$ and $\|A_\xi P_\xi A^{-1}\|_{\mathscr{L}(X)}$ are bounded uniformly in ξ.

($\mathrm{Sp}_\xi 2$) For some $\hat\beta \in [\beta, 1)$, $\|\cdot\|_{Z_\xi} \leqslant \hat D \|A_\xi^{\hat\beta} \cdot\|_{X_\xi}$ with some constant $\hat D$ independent of ξ.

Utilizing the implicit–explicit Euler scheme with stepsize $h > 0$, we obtain the fully discrete approximation to (2.1)

$$\begin{cases} \dfrac{\hat U_n - \hat U_{n-1}}{h} + A_\xi \hat U_n = F_\xi(\hat U_{n-1}), \qquad n = 1, 2, \ldots, N, \\ \hat U_0 = P_\xi U_0, \end{cases} \tag{2.3}$$

where N is a positive integer such that $Nh \leqslant T$.

The results are stated as follows.

Theorem 2.1. *Assume* ($\mathrm{A}_\xi 1$), ($\mathrm{Sp}_\xi 1$–2), (In) *and* (F), *and fix* $h_0 > 0$ *and* $\xi_0 > 0$ *arbitrarily. Then, for any* $0 < h < h_0$ *and* $0 < \xi < \xi_0$, *Eq.* (2.3) *possesses a unique global solution* $\hat{\mathscr{U}} = [\hat U_0, \hat U_1, \ldots, \hat U_N]$ *on the interval* $[0, T]$, *where* $N \leqslant T/h$. *Moreover,* $\hat{\mathscr{U}}$ *satisfies*

$$\max_{n=0,1,\ldots,N_{U_0}} \{\|\hat U_n\|_{X_\xi} + \|A_\xi \hat U_n\|_{X_\xi}\} \leqslant C_{U_0}.$$

Here, N_{U_0} *is a positive integer such that* $N_{U_0} h \leqslant T_{U_0}$, T_{U_0} *and* C_{U_0} *are positive constants determined by* $\|U_0\|_{\mathscr{D}}$.

Theorem 2.2. *Assume* (A), (F), (In), ($\mathrm{A}_\xi 1$–3) *and* ($\mathrm{Sp}_\xi 1$–2). *Let* U *be a solution to* (2.1) *such that* $U \in \mathscr{C}^2([0, T_{U_0}]; X) \cap \mathscr{C}^1([0, T_{U_0}]; \mathscr{D}(A))$. *Then, the errors are estimated by*

$$\max_{n=0,1,\ldots,N} \|\hat U_n - U(t_n)\|_Z \leqslant C_{U_0}[\|(1 - P_\xi)U\|_{\mathscr{C}([0,T_{U_0}];Z)} + \sigma^{-1} Q_\xi^{1-\hat\beta} \|U\|_{\mathscr{C}^\sigma([0,T_{U_0}];\mathscr{D})}$$
$$+ h(\|U''\|_{\mathscr{C}([0,T_{U_0}];X)} + \|U'\|_{\mathscr{C}([0,T_{U_0}];\mathscr{D})})],$$

where $N \leqslant S/h$. *Here,* $\sigma > 0$ *is any exponent and the constant* $C_{U_0} > 0$ *depends on* $\|U_0\|_{\mathscr{D}}$ *and* $\|U\|_{\mathscr{C}([0,T_{U_0}];Z)}$.

Sketch of the proof of the theorems. We can easily see that the solution of (2.3) is given by

$$\hat U_n = (1 + hA_\xi)^{-n} \hat U_0 + h \sum_{\ell=0}^{n-1} (1 + hA_\xi)^{-(n-\ell)} F_\xi(\hat U_\ell), \quad n = 0, 1, \ldots, N. \tag{2.4}$$

The family $\{(1 + hA_\xi)^{-n}\}_{n \geqslant 0}$ of the powers of resolvent works as the discrete version of semigroup $\{\mathrm{e}^{-tA_\xi}\}_{t \geqslant 0}$. The desired estimates can be obtained by the similar discussion in [5–8]. We omit the detail here. □

3. Variational approach

Let $\mathscr{H}$ and $\mathscr{V}$ be two separable Hilbert spaces with dense and compact embedding $\mathscr{V} \subset \mathscr{H}$. Identifying $\mathscr{H}$ and its dual $\mathscr{H}'$ and denoting the dual space of $\mathscr{V}$ by $\mathscr{V}'$, we have $\mathscr{V} \subset \mathscr{H} \equiv \mathscr{H}' \subset \mathscr{V}'$.

Consider the Cauchy problem

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t < T, \\ U(0) = U_0 \end{cases} \tag{3.1}$$

in $\mathscr{V}'$. Here, A is a bounded linear operator from $\mathscr{V}$ to $\mathscr{V}'$ and is a self-adjoint operator in $\mathscr{H}$ which is defined by a symmetric bilinear form $\alpha(\cdot,\cdot)$ on $\mathscr{V}$. $F(\cdot)$ is a continuous operator from $\mathscr{V}$ to $\mathscr{V}'$. $U_0 \in \mathscr{H}$ is an initial value. $U = U(t)$ is the unknown function.

We make the following assumptions:

(A) $\alpha(\cdot,\cdot)$ satisfies

$$\begin{cases} |\alpha(U,V)| \leqslant \alpha_0^{-1}\|U\|_{\mathscr{V}}\|V\|_{\mathscr{V}}, \quad U,V \in \mathscr{V}, \\ \alpha(U,U) \geqslant \alpha_0\|U\|_{\mathscr{V}}^2, \quad U \in \mathscr{V} \end{cases}$$

with some constant $\alpha_0 > 0$;

(F1) $F(\cdot)$ satisfies

$$\|F(U)\|_{\mathscr{V}'} \leqslant \zeta\|U\|_{\mathscr{V}} + \phi_\zeta(\|U\|_{\mathscr{H}}), \quad U \in \mathscr{V}$$

with arbitrary number $\zeta > 0$ and some continuous increasing function $\phi_\zeta(\cdot)$ depending on ζ;

(F2) $F(\cdot)$ satisfies

$$\|F(U) - F(V)\|_{\mathscr{V}'} \leqslant \zeta\|U - V\|_{\mathscr{V}} + (1 + \|U\|_{\mathscr{V}} + \|V\|_{\mathscr{V}}) \times\psi_\zeta(\|U\|_{\mathscr{H}} + \|V\|_{\mathscr{H}})\|U - V\|_{\mathscr{H}}, \quad U,V \in \mathscr{V}$$

with arbitrary number $\zeta > 0$ and some continuous increasing function $\psi_\zeta(\cdot)$ depending on ζ;

(In) U_0 is in $\mathscr{V}$.

It is already known that, under these assumptions, Eq. (3.1) possesses a unique local solution

$$U \in H^1(0,T_{U_0};\mathscr{V}') \cap \mathscr{C}([0,T_{U_0}];\mathscr{H}) \cap L^2(0,T_{U_0};\mathscr{V}),$$

where $T_{U_0} > 0$ is a constant determined by $\|U_0\|_{\mathscr{H}}$, see [9–11].

We can also write (3.1) in the weak form

$$\begin{cases} \left\langle \dfrac{dU}{dt}, W \right\rangle_{\mathscr{V}'\times\mathscr{V}} + \alpha(U,W) = \langle F(U), W\rangle_{\mathscr{V}'\times\mathscr{V}}, \quad W \in \mathscr{V},\ 0 < t < T, \\ U(0) = U_0. \end{cases} \tag{3.2}$$

Let $\{\mathscr{V}_\xi\}_{\xi>0}$ be a family of finite-dimensional subspaces of $\mathscr{V}$. For $\xi > 0$, $P_\xi : \mathscr{V}' \to \mathscr{V}_\xi$ is the projection operator defined by $\langle P_\xi V, \hat{W}\rangle_{\mathscr{V}'\times\mathscr{V}} = \langle V, \hat{W}\rangle_{\mathscr{V}'\times\mathscr{V}}$ for $V \in \mathscr{V}'$ and $\hat{W} \in \mathscr{V}_\xi$.

Then the Galerkin approximation to (3.2) in $\mathscr{V}_\xi$ is given by

$$\begin{cases} \left\langle \dfrac{d\hat{U}}{dt}, \hat{W} \right\rangle_{\mathscr{V}'\times\mathscr{V}} + \alpha(\hat{U},\hat{W}) = \langle F(\hat{U}), \hat{W}\rangle_{\mathscr{V}'\times\mathscr{V}}, \quad \hat{W} \in \mathscr{V}_\xi,\ 0 < t < T, \\ \hat{U}(0) = P_\xi U_0. \end{cases} \tag{3.3}$$

We assume the following condition:

(Sp_ξ) The norms $\|P_\xi\|_{\mathscr{L}(\mathscr{H})}$ and $\|P_\xi\|_{\mathscr{L}(\mathscr{V})}$ are bounded by a positive constant $\hat{M}_P$ independent of ξ.

Utilizing the implicit–explicit Euler scheme with stepsize $h > 0$, we obtain the fully discrete approximation to (3.1)

$$\begin{cases} \left\langle \dfrac{\hat{U}_n - \hat{U}_{n-1}}{h}, \hat{W} \right\rangle_{\mathscr{V}' \times \mathscr{V}} + \alpha(\hat{U}_n, \hat{W}) = \langle F(\hat{U}_{n-1}), \hat{W} \rangle_{\mathscr{V}' \times \mathscr{V}}, \quad \hat{W} \in \mathscr{V}_\xi, \ n = 1, 2, \ldots, N, \\ \hat{U}_0 = P_\xi U_0, \end{cases} \tag{3.4}$$

where N is a positive integer such that $Nh \leqslant T$.

The results are stated as follows.

Theorem 3.1. *Assume* (A), (F1), (In) *and* (Sp_ξ), *and fix* $h_0 > 0$ *and* $\xi_0 > 0$ *arbitrarily. Then, for any* $0 < h < h_0$ *and* $0 < \xi < \xi_0$, *Eq.* (3.4) *possesses a unique global solution* $\hat{\mathscr{U}} = [\hat{U}_0, \hat{U}_1, \ldots, \hat{U}_N]$ *on the interval* $[0, T]$, *where* $N = [T/h]$. *Moreover,* $\hat{\mathscr{U}}$ *satisfies*

$$\max_{n=0,1,\ldots,N_{U_0}} \|\hat{U}_n\|^2_{\mathscr{H}} + h \sum_{n=0}^{N_{U_0}} \|\hat{U}_n\|^2_{\mathscr{V}} \leqslant C_{U_0}. \tag{3.5}$$

Here, N_{U_0} *is a positive integer such that* $N_{U_0} h \leqslant T_{U_0}$, T_{U_0} *and* C_{U_0} *are positive constants determined by* $\|U_0\|_{\mathscr{V}}$.

Theorem 3.2. *Assume* (A), (F1), (F2), (In) *and* (In_ξ). *Let* U *be a solution to* (3.1) *such that* $U \in H^1(0, T_{U_0}; \mathscr{V}) \cap H^2(0, T_{U_0}; \mathscr{V}')$. *Then the errors* $\hat{U}_n - U(t_n)$, $t_n = nh$, *are estimated by*

$$\begin{aligned} &\max_{n=0,1,\ldots,N_{U_0}} \|\hat{U}_n - U(t_n)\|^2_{\mathscr{H}} + h \sum_{n=0}^{N_{U_0}} \|\hat{U}_n - U(t_n)\|^2_{\mathscr{V}} \\ &\quad \leqslant C_U[\|U_{0\xi} - U_0\|^2_{\mathscr{H}} + h\|U_{0\xi} - U_0\|^2_{\mathscr{V}} + \|(1 - P_\xi)U\|^2_{\mathscr{C}([0,T_{U_0}];\mathscr{H})} \\ &\qquad + \|(1 - P_\xi)U\|^2_{L^2(0,T_{U_0};\mathscr{V})} + h^2\|U'\|^2_{L^2(0,T_{U_0};\mathscr{V})} + h^2\|U''\|^2_{L^2(0,T_{U_0};\mathscr{V}')}], \end{aligned} \tag{3.6}$$

where the constant $C_U > 0$ *depends on* $\|U(\cdot)\|_{\mathscr{C}([0,T_{U_0}];\mathscr{V})}$.

Proof of Theorem 3.1. We first rewrite the first equation of (3.4) as

$$\langle \hat{U}_n, \hat{W} \rangle_{\mathscr{V}' \times \mathscr{V}} + h\alpha(\hat{U}_n, \hat{W}) = \langle \hat{U}_{n-1}, \hat{W} \rangle_{\mathscr{V}' \times \mathscr{V}} + h\langle F(\hat{U}_{n-1}), \hat{W} \rangle_{\mathscr{V}' \times \mathscr{V}}, \quad \hat{W} \in \mathscr{V}_\xi. \tag{3.7}$$

Step 1: *Existence of solution.*

Let $\{\hat{W}_1, \ldots, \hat{W}_{\ell_\xi}\}$ be a basis of $\mathscr{V}_\xi$, where $\ell_\xi = \dim \mathscr{V}_\xi$. Then $\hat{U}_n$ can be written as

$$\hat{U}_n = \sum_{i=1}^{\ell_\xi} U_{n,i} \hat{W}_i, \quad \boldsymbol{U}_n = [U_{n,i}].$$

Eq. (3.7) is equivalent to the linear algebraic system

$$(\mathbb{M} + h\mathbb{A})\boldsymbol{U}_n = \mathbb{M}\boldsymbol{U}_{n-1} + h\mathbb{F}(\boldsymbol{U}_{n-1}), \tag{3.8}$$

where $\mathbb{M} = [\langle \hat{W}_j, \hat{W}_i\rangle_{\mathscr{V}'\times\mathscr{V}}]$ and $\mathbb{A} = [\alpha(\hat{W}_j, \hat{W}_i)]$ are $\ell_\xi \times \ell_\xi$-matrices and $\mathbb{F}(\cdot)$ is a function on $\mathbb{R}^{\ell_\xi}$ given by $\mathbb{F}(\boldsymbol{U}) = [\langle F(\sum_j U_j\hat{W}_j), \hat{W}_i\rangle_{\mathscr{V}'\times\mathscr{V}}]$. It is easily seen that $\mathbb{M} + h\mathbb{A}$ is positive definite. As a direct consequence, (3.8) has a unique solution $\boldsymbol{U}_n$ for arbitrary n. Hence, (3.7) possesses a global unique solution $\hat{\mathscr{U}} = [\hat{U}_0, \hat{U}_1, \ldots, \hat{U}_N]$, where $Nh \leqslant T$.

Step 2: *A priori estimate*.

Taking $\hat{W} = \hat{U}_n$ in Eq. (3.7), we have

$$\|\hat{U}_n\|^2_{\mathscr{H}} - \|\hat{U}_{n-1}\|^2_{\mathscr{H}} + h\alpha_0\|\hat{U}_n\|^2_{\mathscr{V}} \leqslant h\frac{\alpha_0}{2}\|\hat{U}_{n-1}\|^2_{\mathscr{V}} + h\tilde{\phi}(\|\hat{U}_{n-1}\|^2_{\mathscr{H}}).$$

Here $\tilde{\phi}(\cdot)$ is a non-decreasing locally Lipschitz continuous function satisfying $\tilde{\phi}(r) \geqslant (2/\alpha_0)\phi_{\alpha_0/2}(\sqrt{r})^2$. Summing up this inequality, we have

$$\|\hat{U}_n\|^2_{\mathscr{H}} + h\frac{\alpha_0}{2}\sum_{k=1}^{n}\|\hat{U}_k\|^2_{\mathscr{V}} \leqslant \|\hat{U}_0\|^2_{\mathscr{H}} + h\frac{\alpha_0}{2}\|\hat{U}_0\|^2_{\mathscr{V}} + h\sum_{k=0}^{n-1}\tilde{\phi}(\|\hat{U}_k\|^2_{\mathscr{H}}).$$

Denoting the left side member by X_n and comparing it to the solution $y(t)$ of the differential equation

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = \tilde{\phi}(y), \quad 0 \leqslant t \leqslant T, \\ y(0) = \left(1 + h_0\dfrac{\alpha_0}{2}\right)\hat{M}_P(\|U_0\|_{\mathscr{H}} + \|U_0\|_{\mathscr{V}})^2, \end{cases} \tag{3.9}$$

we can verify that

$$\max_{k=0,1,\ldots,N_{U_0}}\|\hat{U}_k\|^2_{\mathscr{H}} + h\,\frac{\alpha_0}{2}\sum_{k=1}^{N_{U_0}}\|\hat{U}_k\|^2_{\mathscr{V}} = \max_{k=0,1,\ldots,N_{U_0}} X_k \leqslant y(T_{U_0}).$$

Here T_{U_0} is the lifespan of $y(t)$, and $N_{U_0} = [T_{U_0}/h]$. Thus we complete the proof. □

Outline of the proof of Theorem 3.2. We denote here $E_n = \hat{U}_n - U(t_n)$, $\tilde{E}_n = P_\xi E_n$ and $U_n = U(t_n)$. Then we can easily see that $E_n = P_\xi E_n + (1 - P_\xi)E_n = \tilde{E}_n - (1 - P_\xi)U(t_n)$, and $\langle E_n, \hat{W}\rangle_{\mathscr{V}'\times\mathscr{V}} = \langle \tilde{E}_n, \hat{W}\rangle_{\mathscr{V}'\times\mathscr{V}}$, $\hat{W} \in \mathscr{V}_\xi$.

We introduce the error e_n defined by $e_n = U(t_n) - U(t_{n-1}) - hU'(t_n)$. Then, from (3.2),

$$\left\langle \frac{U_n - U_{n-1}}{h}, W\right\rangle_{\mathscr{V}'\times\mathscr{V}} + \alpha(U_n, W) = \langle F(U_n), W\rangle_{\mathscr{V}'\times\mathscr{V}} + \left\langle \frac{e_n}{h}, W\right\rangle_{\mathscr{V}'\times\mathscr{V}}, \quad W \in \mathscr{V}.$$

Subtracting this equation from (3.4), we obtain the error equation

$$\begin{cases} \left\langle \dfrac{E_n - E_{n-1}}{h}, \hat{W}\right\rangle_{\mathscr{V}'\times\mathscr{V}} + \alpha(E_n, \hat{W}) = \left\langle F(\hat{U}_{n-1}) - F(U_n) - \dfrac{e_n}{h}, \hat{W}\right\rangle_{\mathscr{V}'\times\mathscr{V}}, \quad \hat{W} \in \mathscr{V}_\xi, \\ E_0 = U_{0\xi} - U_0. \end{cases}$$

In a similar way to the above proof, taking $\hat{W} = \tilde{E}_n$ and summing up, we can verify the energy inequality on E_n

$$\|E_n\|_{\mathscr{H}}^2 + \frac{3\alpha_0}{4} h \sum_{k=1}^{n} \|E_k\|_{\mathscr{V}}^2 \leqslant C\varepsilon_n + h \sum_{k=0}^{n-1} (1 + \|\hat{U}_k\|_{\mathscr{V}}^2 + \|U_k\|_{\mathscr{V}}^2)\tilde{\psi}(\|\hat{U}_k\|_{\mathscr{H}}^2 + \|U_k\|_{\mathscr{H}}^2)\|E_k\|_{\mathscr{H}}^2.$$

Here $\tilde{\psi}(\cdot)$ is a non-decreasing locally Lipschitz continuous function, and

$$\begin{aligned}\varepsilon_n = {} & \|E_0\|_{\mathscr{H}}^2 + h\|E_0\|_{\mathscr{V}}^2 + \max_{k=0,1,\ldots,n} \|(1-P_\xi)U_k\|_{\mathscr{H}}^2 + h\sum_{k=1}^{n} \|(1-P_\xi)U_k\|_{\mathscr{V}}^2 \\ & + h\sum_{k=1}^{n} \|e_k/h\|_{\mathscr{V}'}^2 + h\sum_{k=1}^{n} \|U_k - U_{k-1}\|_{\mathscr{V}}^2 \\ & + h\sum_{k=1}^{n} (1 + \|U_{k-1}\|_{\mathscr{V}}^2 + \|U_k\|_{\mathscr{V}}^2)\tilde{\psi}(\|U_{k-1}\|_{\mathscr{H}}^2 + \|U_k\|_{\mathscr{H}}^2)\|U_{k-1} - U_k\|_{\mathscr{H}}^2.\end{aligned}$$

With the aid of the Gronwall inequality for difference equations and a priori estimates of $\hat{\mathscr{U}}$, we have

$$\|E_n\|_{\mathscr{H}}^2 + \frac{3\alpha_0}{4} h \sum_{k=1}^{n} \|E_k\|_{\mathscr{V}}^2 \leqslant C_U \varepsilon_n.$$

Straightforward estimates for ε_n by the derivatives of $U(\cdot)$ lead to the desired result. We omit the detail here because the estimation procedure is very simple but very lengthy. □

4. Concluding remarks

We have studied in this article two approximation problems for evolution equations in different abstract spaces. Finally we comment on the similarity and difference between these two problems.

Let $X = \mathscr{V}'$ and $\mathscr{D} = \mathscr{V}$, then there are much correspondence between the setting and results in Section 2 and those in Section 3 as follows.

(1) The operator A in Section 3 is also the negative generator of an analytic semigroup on $\mathscr{V}'$ (see e.g. [12]), and satisfies the condition (A) with arbitrarily small φ in Section 2.
(2) The assumption (F) in Section 2 leads (F1–2) in Section 3 in some cases. In fact, if the function $p(\cdot)$ in Section 2 is polynomial, and if Z is an interpolation of $\mathscr{H}$ and $\mathscr{V}$, the operator F satisfying (F) in Section 2 satisfies also (F1–2) in Section 3.
(3) If we define the approximate operator A_ξ by

$$\langle A_\xi \hat{V}, \hat{W} \rangle_{\mathscr{V}' \times \mathscr{V}} = \alpha(\hat{V}, \hat{W}),$$

then we can obtain scheme (2.3) from scheme (3.4).

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